

HIGHER ORDER SPT FUNCTIONS FOR OVERPARTITIONS, OVERPARTITIONS WITH SMALLEST PART EVEN, AND PARTITIONS WITHOUT REPEATED ODD PARTS

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ABSTRACT. We consider the symmetrized moments of three ranks and cranks, similar to the work of Garvan in [17] for the rank and crank of a partition. By using Bailey pairs and elementary rearrangements, we are able to find useful expressions for these moments. We then deduce inequalities between the corresponding ordinary moments. In particular we prove that the crank moment for overpartitions is always larger than the rank moment for overpartitions, $\overline{M}_{2k}(n) > \overline{N}_{2k}(n)$; with recent asymptotics this was known to hold for sufficiently large values of n for each fixed k . Lastly we provide higher order spt functions for overpartitions, overpartitions with smallest part even, and partitions with smallest part even and no repeated odds.

1. INTRODUCTION AND STATEMENT OF RESULTS

Here we consider certain rank and crank moments for partition like functions. We recall a partition of n is a non-increasing sequence of positive integers that sum to n . We denote the number of partitions of n by $p(n)$.

Next an overpartition of n is a partition of n in which the first occurrence of a part may be overlined. We denote the number of overpartitions of n by $\overline{p}(n)$. Thus while $p(4) = 5$ since the partitions of 4 are only 4, $3 + 1$, $2 + 2$, $2 + 1 + 1$ and $1 + 1 + 1 + 1$, we have instead $\overline{p}(4) = 14$ since the overpartitions of 4 are 4, $\overline{4}$, $3 + 1$, $3 + \overline{1}$, $\overline{3} + 1$, $\overline{3} + \overline{1}$, $2 + 2$, $\overline{2} + 2$, $2 + 1 + 1$, $2 + \overline{1} + 1$, $\overline{2} + 1 + 1$, $\overline{2} + \overline{1} + 1$, $1 + 1 + 1 + 1$, and $\overline{1} + 1 + 1 + 1$.

In [3] Andrews defined $\text{spt}(n)$ to be the total number of occurrences of the smallest part in each partition of n . In [9] Bringmann, Lovejoy, and Osburn defined $\overline{\text{spt}}(n)$ as the number of smallest parts in the overpartitions of n and $\overline{\text{spt}2}(n)$ to be the number of smallest parts in the overpartitions of n with smallest part even. We use the convention of only counting the smallest parts of the overpartitions where the smallest part is not overlined. In [1] Ahlgren, Bringmann, and Lovejoy defined $\text{M2spt}(n)$ to be the number of smallest parts in the partitions of n without repeated odd parts and with smallest part even. Thus $\text{spt}(4) = 10$, $\overline{\text{spt}}(4) = 13$, $\overline{\text{spt}2}(4) = 3$, and $\text{M2spt}(4) = 3$.

We recall the rank of a partition is the largest parts minus the number of parts. The crank of a partition is the largest part if there are no ones and otherwise is the number of parts larger than the number of ones minus the number of ones. The first point of interest of the rank and crank of a partition is that the rank gives a combinatorial explanation of the well known congruences $p(5n + 4) \equiv 0 \pmod{5}$ and $p(7n + 5) \equiv 0 \pmod{7}$ and the crank gives a combinatorial explanation of $p(5n + 4) \equiv 0 \pmod{5}$, $p(7n + 5) \equiv 0 \pmod{7}$, and $p(11n + 6) \equiv 0 \pmod{11}$. However, the rank and crank have proved to have many further uses. We let $N(m, n)$ denote the number of partitions of n with rank m and $M(m, n)$ denote the number of partitions of n with crank m . After suitably altering the interpretations for $n = 0$ and $n = 1$, one has that

$$C(z, q) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M(m, n) z^m q^n = \frac{(q; q)_{\infty}}{(zq, z^{-1}q; q)_{\infty}}.$$

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For the rank we have

$$\begin{aligned} R(z, q) &= \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N(m, n) z^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq, z^{-1}q; q)_n} \\ &= \frac{1}{(q; q)_{\infty}} \left[1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n(3n+1)/2} (1+q^n)}{(1-zq^n)(1-z^{-1}q^n)} \right]. \end{aligned}$$

Here and throughout the rest of this paper we are using the standard product notation,

$$\begin{aligned} (a; q)_n &= \prod_{j=0}^{n-1} (1 - aq^j), \\ (a; q)_{\infty} &= \prod_{j=0}^{\infty} (1 - aq^j), \\ (a_1, \dots, a_k; q)_n &= (a_1; q)_n \dots (a_k; q)_n, \\ (a_1, \dots, a_k; q)_{\infty} &= (a_1; q)_{\infty} \dots (a_k; q)_{\infty}. \end{aligned}$$

We can now introduce the rank and crank moments

$$\begin{aligned} N_k(n) &= \sum_{m=-\infty}^{\infty} m^k N(m, n), \\ M_k(n) &= \sum_{m=-\infty}^{\infty} m^k M(m, n). \end{aligned}$$

Both of these sums are actually finite since $N(m, n) = M(m, n) = 0$ for $|m| > n$. These moments were first considered by Atkin and Garvan in [7]. By Andrews [3] $\text{spt}(n) = np(n) - \frac{1}{2}N_2(n)$ and by Dyson [14] $np(n) = \frac{1}{2}M_2(n)$, thus

$$\text{spt}(n) = \frac{1}{2}M_2(n) - \frac{1}{2}N_2(n).$$

We then see a useful way to study smallest parts functions is to consider the related rank and crank moments. We next explain the different ranks and cranks we will use.

As in [9], for an overpartition π of n we define a residual crank of π by the crank of the subpartition of π consisting of the non-overlined parts of π . We let $\overline{M}(m, n)$ denote the number of overpartitions of n with this residual crank equal to m . The generating function for $\overline{M}(m, n)$ is then given by

$$\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \overline{M}(m, n) z^m q^n = \frac{(-q; q)_{\infty} (q; q)_{\infty}}{(zq; q)_{\infty} (z^{-1}q; q)_{\infty}}. \quad (1.1)$$

Of course this interpretation is not quite correct, as $\frac{(q; q)_{\infty}}{(zq, z^{-1}q; q)_{\infty}}$ does not agree at q for the crank of the partition consisting of a single one. Thus the interpretation of this residual crank is not quite correct for overpartitions whose non-overlined parts consist of a single one. However, this is the generating function we must use.

As in [9] and others, for an overpartition π of n we define the Dyson rank of π to be the largest part minus the number of parts of π . We let $\overline{N}(m, n)$ denote the number of overpartitions of n with Dyson rank equal to m . As in Proposition 1.1 and the proof of Proposition 3.2 of [19], the generating function for $\overline{N}(m, n)$ is given by

$$\overline{R}(z, q) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \overline{N}(m, n) z^m q^n = \sum_{n=0}^{\infty} \frac{(-1; q)_n q^{n(n+1)/2}}{(zq; q)_n (z^{-1}q; q)_n} \quad (1.2)$$

$$= \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \left[1 + 2 \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n^2+n}}{(1-zq^n)(1-z^{-1}q^n)} \right]. \quad (1.3)$$

The second equality is obtained by Watson's transformation.

We use another residual crank that was defined in [15]. For a partition π of n with distinct odd parts we take the crank of the partition $\frac{\pi_e}{2}$ obtained by taking the subpartition π_e , of the even parts of π , and halving each part of π_e . We let $M2(m, n)$ denote the number of partitions π of n with distinct odd parts and such that the partition $\frac{\pi_e}{2}$ has crank m . Then the generating function for $M2(m, n)$ is given by

$$\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M2(m, n) z^m q^n = \frac{(-q; q^2)_{\infty} (q^2; q^2)_{\infty}}{(zq^2; q^2)_{\infty} (z^{-1}q^2; q^2)_{\infty}}. \quad (1.4)$$

Of course this interpretation is not quite correct, here it fails for partitions with distinct odd parts whose only even parts are a single two.

We recall the M_2 -rank of a partition π without repeated odd parts is given by

$$M_2\text{-rank} = \left\lceil \frac{l(\pi)}{2} \right\rceil - \#(\pi), \quad (1.5)$$

where $l(\pi)$ is the largest part of π and $\#(\pi)$ is the number of parts of π . The M_2 -rank was introduced by Berkovich and Garvan in [8]. We let $N2(m, n)$ denote the number of partitions of n with distinct odd parts and M_2 -rank m . By Lovejoy and Osburn [21] the generating function for $N2(m, n)$, which we further rearrange as in [15], is given by

$$R2(z, q) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N2(m, n) z^m q^n = \sum_{n=0}^{\infty} q^{n^2} \frac{(-q; q^2)_n}{(zq^2; q^2)_n (z^{-1}q^2; q^2)_n} \quad (1.6)$$

$$= \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \left[1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n(2n+1)}(1+q^{2n})}{(1-zq^{2n})(1-z^{-1}q^{2n})} \right]. \quad (1.7)$$

We also use the second residual crank from [9]. For an overpartition π of n we take the crank of the partition $\frac{\pi_e}{2}$ obtained by taking the subpartition π_e , of the even non-overlined parts of π , and halving each part of π_e . We let $\overline{M2}(m, n)$ denote the number of overpartitions π of n and such that the partition $\frac{\pi_e}{2}$ has crank m . Then the generating function for $\overline{M2}$ is given by

$$\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \overline{M2}(m, n) z^m q^n = \frac{(-q; q)_{\infty} (q^2; q^2)_{\infty}}{(q; q^2)_{\infty} (zq^2; q^2)_{\infty} (z^{-1}q^2; q^2)_{\infty}}. \quad (1.8)$$

Again this interpretation fails for overpartitions whose only even non-overlined parts are a single two.

Lastly, we use the M_2 -rank of a overpartition π . This rank is given by

$$M_2\text{-rank} = \left\lceil \frac{l(\pi)}{2} \right\rceil - \#(\pi) + \#(\pi_o) - \chi(\pi), \quad (1.9)$$

where π_o is the subpartition consisting of the odd non-overlined parts, and $\chi(\pi) = 1$ if the largest part of π is odd and non-overlined and $\chi(\pi) = 0$ otherwise. The M_2 -rank for overpartitions was introduced by Lovejoy in [20]. We let $\overline{N2}(m, n)$ denote the number of overpartitions of n with M_2 -rank m . Lovejoy found the generating function for $\overline{N2}$ is given by

$$\overline{R2}(z, q) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \overline{N2}(m, n) z^m q^n = \sum_{n=0}^{\infty} q^n \frac{(-1; q)_{2n}}{(zq^2; q^2)_n (z^{-1}q^2; q^2)_n} \quad (1.10)$$

$$= \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \left[1 + 2 \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n^2+2n}}{(1-zq^{2n})(1-z^{-1}q^{2n})} \right]. \quad (1.11)$$

We then have the various rank and crank moments:

$$N2_k(n) = \sum_{m \in \mathbb{Z}} m^k N2(m, n),$$

$$\begin{aligned}
M2_k(n) &= \sum_{m \in \mathbb{Z}} m^k M2(m, n), \\
\overline{N}_k(n) &= \sum_{m \in \mathbb{Z}} m^k \overline{N}(m, n), \\
\overline{M}_k(n) &= \sum_{m \in \mathbb{Z}} m^k \overline{M}(m, n), \\
\overline{N2}_k(n) &= \sum_{m \in \mathbb{Z}} m^k \overline{N2}(m, n), \\
\overline{M2}_k(n) &= \sum_{m \in \mathbb{Z}} m^k \overline{M2}(m, n).
\end{aligned}$$

Rather than immediately working with these moments, it has proved fruitful to consider a symmetrized version (for examples of this see [2], [10], [13], [17], and [22]) . We let

$$\begin{aligned}
\eta_k(n) &= \sum_{m \in \mathbb{Z}} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} N(m, n), \\
\mu_k(n) &= \sum_{m \in \mathbb{Z}} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} M(m, n), \\
\eta_{2k}(n) &= \sum_{m \in \mathbb{Z}} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} N2(m, n), \\
\mu_{2k}(n) &= \sum_{m \in \mathbb{Z}} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} M2(m, n), \\
\overline{\eta}_k(n) &= \sum_{m \in \mathbb{Z}} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} \overline{N}(m, n), \\
\overline{\mu}_r(n) &= \sum_{m \in \mathbb{Z}} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} \overline{M}(m, n), \\
\overline{\eta}_{2k}(n) &= \sum_{m \in \mathbb{Z}} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} \overline{N2}(m, n), \\
\overline{\mu}_{2k}(n) &= \sum_{m \in \mathbb{Z}} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} \overline{M2}(m, n).
\end{aligned}$$

We note all of these series are actually finite since for each rank and crank, for fixed n there are no partitions or overpartitions with rank or crank past n or $-n$. Also in all cases the ranks and cranks are symmetric in m and $-m$. Because of this symmetry, one can deduce the moments for odd k vanish, hence we'll only consider the even moments in this paper. To handle odd k various authors, such as in [4], [5], [11], and [18], have considered sums over $m \geq 1$ instead of all m , these series are denoted by a $+$ superscript.

In Theorem 4.3 of [17], Garvan found how to switch between η_{2k} and N_{2k} and between μ_{2k} and M_{2k} . Using this process we have

$$\begin{aligned}
\eta_{2k}(n) &= \frac{1}{(2k)!} \sum_{m \in \mathbb{Z}} g_k(m) N2(m, n), \\
\mu_{2k}(n) &= \frac{1}{(2k)!} \sum_{m \in \mathbb{Z}} g_k(m) M2(m, n), \\
\overline{\eta}_{2k}(n) &= \frac{1}{(2k)!} \sum_{m \in \mathbb{Z}} g_k(m) \overline{N}(m, n), \\
\overline{\mu}_{2k}(n) &= \frac{1}{(2k)!} \sum_{m \in \mathbb{Z}} g_k(m) \overline{M}(m, n),
\end{aligned}$$

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$$\begin{aligned}
\overline{\eta 2}_{2k}(n) &= \frac{1}{(2k)!} \sum_{m \in \mathbb{Z}} g_k(m) \overline{N 2}(m, n), \\
\overline{\mu 2}_{2k}(n) &= \frac{1}{(2k)!} \sum_{m \in \mathbb{Z}} g_k(m) \overline{M 2}(m, n), \\
N 2_{2k}(n) &= \sum_{j=1}^k (2j)! S^*(k, j) \eta 2_{2j}(n), \\
M 2_{2k}(n) &= \sum_{j=1}^k (2j)! S^*(k, j) \mu 2_{2j}(n), \\
\overline{N 2}_{2k}(n) &= \sum_{j=1}^k (2j)! S^*(k, j) \overline{\eta 2}_{2j}(n), \\
\overline{M 2}_{2k}(n) &= \sum_{j=1}^k (2j)! S^*(k, j) \overline{\mu 2}_{2j}(n), \\
\overline{N 2}_{2k}(n) &= \sum_{j=1}^k (2j)! S^*(k, j) \overline{\eta 2}_{2j}(n), \\
\overline{M 2}_{2k}(n) &= \sum_{j=1}^k (2j)! S^*(k, j) \overline{\mu 2}_{2j}(n).
\end{aligned}$$

Here

$$g_k(x) = \prod_{j=0}^{k-1} (x^2 - j^2)$$

and the sequence $S^*(n, k)$ is defined recursively by $S^*(n+1, k) = S^*(n, k-1) + k^2 S^*(n, k)$ and the boundary conditions $S^*(1, 1) = 1$, $S^*(n, k) = 0$ for $k \leq 0$ or $k > n$.

In [12] Bringmann, Mahlburg, and Rhoades derived asymptotics for M_{2k} , N_{2k} , and $M_{2k} - N_{2k}$. In particular this showed that for each k , for sufficiently large n one has $M_{2k}(n) - N_{2k}(n) > 0$. In [17] it was proved that indeed $M_{2k}(n) - N_{2k}(n) > 0$ for all k and n . Recently Zapata Rolon [24] has worked out the asymptotics for \overline{M}_k^+ , \overline{N}_k^+ , and $\overline{M}_k^+ - \overline{N}_k^+$. This similarly gives that for each k , for sufficiently large n one has $\overline{M}_{2k}(n) - \overline{N}_{2k}(n) > 0$. Here we prove that indeed $\overline{M}_{2k}(n) - \overline{N}_{2k}(n) > 0$ for all n .

Next we define

$$\begin{aligned}
\text{M2spt}_k(n) &= \mu 2_{2k}(n) - \eta 2_{2k}(n), \\
\overline{\text{spt}}_k(n) &= \overline{\mu 2}_{2k}(n) - \overline{\eta 2}_{2k}(n), \\
\overline{\text{spt} 2}_k(n) &= \overline{\mu 2}_{2k}(n) - \overline{\eta 2}_{2k}(n),
\end{aligned}$$

Unlike in [15], here $k = 1, 2$ as a subscript in $\overline{\text{spt}}(n)$ does not specify the smallest part being odd or even (for this restriction we are using $\text{spt} 2(n)$).

The purpose of this paper is to find expressions for $\mu 2$, $\eta 2$, $\overline{\mu}$, $\overline{\eta}$, $\overline{\mu 2}$, and $\overline{\eta 2}$; use these expressions to see $\text{M2spt}_k(n)$, $\overline{\text{spt}}_k(n)$, $\overline{\text{spt} 2}_k(n)$ are non-negative; and further use these expressions to see that $M 2(n) > N 2(n)$, $\overline{M}(n) > \overline{N}(n)$, and $\overline{M 2}(n) > \overline{N 2}(n)$. We then give combinatorial interpretations of the $\text{M2spt}_k(n)$, $\overline{\text{spt}}_k(n)$, and $\overline{\text{spt} 2}_k(n)$. We end by proving two congruences for $\overline{\text{spt}}_2(n)$. All the machinery from [17] can be immediately reused for these purposes.

2. THEOREMS AND PROOFS

For $C2(z, q)$, $\overline{C}(z, q)$, and $\overline{C2}(z, q)$ we use that

$$\frac{(q; q)_\infty}{(zq, z^{-1}q; q)_\infty} = \frac{1}{(q; q)_\infty} \left(1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n(n+1)/2}(1+q^n)}{(1-zq^n)(1-z^{-1}q^n)} \right),$$

this is [16, equation (7.15)]. Thus

$$\begin{aligned} C2(z, q) &= \frac{(-q; q^2)_\infty (q^2; q^2)_\infty}{(zq^2; q^2)_\infty (z^{-1}q^2; q^2)_\infty} \\ &= \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \left[1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n(n+1)}(1+q^{2n})}{(1-zq^{2n})(1-z^{-1}q^{2n})} \right] \\ &= \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \left[1 + \sum_{n=1}^{\infty} (-1)^n q^{n(n+1)} \left(\frac{1-z}{1-zq^{2n}} + \frac{1-z^{-1}}{1-z^{-1}q^{2n}} \right) \right] \\ &= \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)}(1-z)}{1-zq^{2n}}. \end{aligned}$$

And similarly we have

$$\begin{aligned} \overline{C}(z, q) &= \frac{(-q; q)_\infty}{(q; q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2}(1-z)}{1-zq^n}, \\ \overline{C2}(z, q) &= \frac{(-q; q)_\infty}{(q; q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)}(1-z)}{1-zq^{2n}}. \end{aligned}$$

We find similar expressions for the ranks.

$$\begin{aligned} R2(z, q) &= \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \left[1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n(2n+1)}(1+q^{2n})}{(1-zq^{2n})(1-z^{-1}q^{2n})} \right] \\ &= \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \left[1 + \sum_{n=1}^{\infty} (-1)^n q^{n(2n+1)} \left(\frac{1-z}{1-zq^{2n}} + \frac{1-z^{-1}}{1-z^{-1}q^{2n}} \right) \right] \\ &= \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(2n+1)}(1-z)}{1-zq^{2n}}. \end{aligned}$$

Next we have,

$$\begin{aligned} \overline{R}(z, q) &= \frac{(-q; q)_\infty}{(q; q)_\infty} \left[1 + 2 \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n^2+n}}{(1-zq^n)(1-z^{-1}q^n)} \right] \\ &= \frac{(-q; q)_\infty}{(q; q)_\infty} \left[1 + 2 \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2+n}}{(1+q^n)} \left(\frac{1-z}{1-zq^n} + \frac{1-z^{-1}}{1-z^{-1}q^n} \right) \right] \\ &= 2 \frac{(-q; q)_\infty}{(q; q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+n}(1-z)}{(1+q^n)(1-zq^n)}. \end{aligned}$$

Similarly we have

$$\overline{R2}(z, q) = 2 \frac{(-q; q)_\infty}{(q; q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+2n}(1-z)}{(1+q^{2n})(1-zq^{2n})}.$$

Using that

$$\left(\frac{\partial}{\partial z}\right)^j \frac{1-z}{1-zq^n} = \frac{-j!(1-q^n)q^{n(j-1)}}{(1-zq^n)^{j+1}},$$

we find that

$$\begin{aligned} C2^{(j)}(z, q) &= \left(\frac{\partial}{\partial z}\right)^j C2(z, q) = \frac{-j!(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{n \neq 0} \frac{(-1)^n q^{n(n-1)+2jn}(1-q^{2n})}{(1-zq^{2n})^{j+1}}, \\ R2^{(j)}(z, q) &= \left(\frac{\partial}{\partial z}\right)^j R2(z, q) = \frac{-j!(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{n \neq 0} \frac{(-1)^n q^{n(2n-1)+2jn}(1-q^{2n})}{(1-zq^{2n})^{j+1}}, \\ \overline{C}^{(j)}(z, q) &= \left(\frac{\partial}{\partial z}\right)^j \overline{C}(z, q) = \frac{-j!(-q; q)_\infty}{(q; q)_\infty} \sum_{n \neq 0} \frac{(-1)^n q^{n(n-1)/2+jn}(1-q^n)}{(1-zq^n)^{j+1}}, \\ \overline{R}^{(j)}(z, q) &= \left(\frac{\partial}{\partial z}\right)^j \overline{R}(z, q) = 2 \frac{-j!(-q; q)_\infty}{(q; q)_\infty} \sum_{n \neq 0} \frac{(-1)^n q^{n^2+jn}(1-q^n)}{(1+q^n)(1-zq^n)^{j+1}}, \\ \overline{C2}^{(j)}(z, q) &= \left(\frac{\partial}{\partial z}\right)^j \overline{C2}(z, q) = \frac{-j!(-q; q)_\infty}{(q; q)_\infty} \sum_{n \neq 0} \frac{(-1)^n q^{n(n-1)+2jn}(1-q^{2n})}{(1-zq^{2n})^{j+1}}, \\ \overline{R2}^{(j)}(z, q) &= \left(\frac{\partial}{\partial z}\right)^j \overline{R2}(z, q) = 2 \frac{-j!(-q; q)_\infty}{(q; q)_\infty} \sum_{n \neq 0} \frac{(-1)^n q^{n^2+2jn}(1-q^{2n})}{(1+q^{2n})(1-zq^{2n})^{j+1}}. \end{aligned}$$

We collect all expressions for the symmetrized moments in one theorem. Some of these have been used and proved before in the various papers about these moments.

Theorem 2.1. *For all $k \geq 1$*

$$\begin{aligned} \sum_{n=1}^{\infty} \mu_{2k}(n)q^n &= \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{n \neq 0} \frac{(-1)^{n+1} q^{n(n+1)+2kn}}{(1-q^{2n})^{2k}} = \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{n \geq 1} \frac{(-1)^{n+1} q^{n(n-1)+2kn}(1+q^{2n})}{(1-q^{2n})^{2k}}, \\ \sum_{n=1}^{\infty} \eta_{2k}(n)q^n &= \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{n \neq 0} \frac{(-1)^{n+1} q^{n(2n+1)+2kn}}{(1-q^{2n})^{2k}} = \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{n \geq 1} \frac{(-1)^{n+1} q^{n(2n-1)+2kn}(1+q^{2n})}{(1-q^{2n})^{2k}}, \\ \sum_{n=1}^{\infty} \overline{\mu}_{2k}(n)q^n &= \frac{(-q; q)_\infty}{(q; q)_\infty} \sum_{n \neq 0} \frac{(-1)^{n+1} q^{n(n+1)/2+kn}}{(1-q^n)^{2k}} = \frac{(-q; q)_\infty}{(q; q)_\infty} \sum_{n \geq 1} \frac{(-1)^{n+1} q^{n(n-1)/2+kn}(1+q^n)}{(1-q^n)^{2k}}, \\ \sum_{n=1}^{\infty} \overline{\eta}_{2k}(n)q^n &= 2 \frac{(-q; q)_\infty}{(q; q)_\infty} \sum_{n \neq 0} \frac{(-1)^{n+1} q^{n^2+n+kn}}{(1+q^n)(1-q^n)^{2k}} = 2 \frac{(-q; q)_\infty}{(q; q)_\infty} \sum_{n \geq 1} \frac{(-1)^{n+1} q^{n^2+kn}}{(1-q^n)^{2k}}, \\ \sum_{n=1}^{\infty} \overline{\mu}_{2k}(n)q^n &= \frac{(-q; q)_\infty}{(q; q)_\infty} \sum_{n \neq 0} \frac{(-1)^{n+1} q^{n(n+1)+2kn}}{(1-q^{2n})^{2k}} = \frac{(-q; q)_\infty}{(q; q)_\infty} \sum_{n \geq 1} \frac{(-1)^{n+1} q^{n(n-1)+2kn}(1+q^{2n})}{(1-q^{2n})^{2k}}, \\ \sum_{n=1}^{\infty} \overline{\eta}_{2k}(n)q^n &= 2 \frac{(-q; q)_\infty}{(q; q)_\infty} \sum_{n \neq 0} \frac{(-1)^{n+1} q^{n^2+2n+2kn}}{(1+q^{2n})(1-q^{2n})^{2k}} = 2 \frac{(-q; q)_\infty}{(q; q)_\infty} \sum_{n \geq 1} \frac{(-1)^{n+1} q^{n^2+2kn}}{(1-q^{2n})^{2k}}. \end{aligned}$$

Proof. We follow the proof for a similar expression in Theorem 2 of [2].

$$\begin{aligned} \sum_{n=1}^{\infty} \mu_{2k}(n)q^n &= \frac{1}{(2k)!} \left(\left(\frac{\partial}{\partial z}\right)^{2k} z^{k-1} C2(z, q) \right) \Big|_{z=1} \\ &= \frac{1}{(2k)!} \sum_{j=0}^{k-1} \binom{2k}{j} (k-1) \dots (k-j) C2^{2k-j}(1, q) \end{aligned}$$

$$\begin{aligned}
&= \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{j=0}^{k-1} \binom{k-1}{j} \sum_{n \neq 0} \frac{(-1)^{n+1} q^{n(n-1)+2n(2k-j)} (1 - q^{2n})}{(1 - q^{2n})^{2k-j+1}} \\
&= \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{n \neq 0} \frac{(-1)^{n+1} q^{n(n-1)+4nk}}{(1 - q^{2n})^{2k}} \sum_{j=0}^{k-1} \binom{k-1}{j} (q^{-2n} (1 - q^{2n}))^j \\
&= \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{n \neq 0} \frac{(-1)^{n+1} q^{n(n-1)+4nk}}{(1 - q^{2n})^{2k}} (1 + q^{-2n} (1 - q^{2n}))^{k-1} \\
&= \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{n \neq 0} \frac{(-1)^{n+1} q^{n(n+1)+2nk}}{(1 - q^{2n})^{2k}} \\
&= \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{n \geq 1} \frac{(-1)^{n+1} q^{n(n-1)+2kn} (1 + q^{2n})}{(1 - q^{2n})^{2k}}.
\end{aligned}$$

We omit the proofs of the other identities, as they are near identical to the above, but with $C2(z, q)$ replaced with $R2(z, q)$, $\overline{C}2(z, q)$, $\overline{R}2(z, q)$, $\overline{C}2(z, q)$, and $\overline{R}2(z, q)$ respectively. \square

We recall two sequences of functions α_n and β_n are a Bailey pair relative to (a, q) if

$$\beta_n = \sum_{r=0}^n \frac{\alpha_n}{(q; q)_{n-r} (aq; q)_{n+r}}.$$

The following is Theorem 3.3 of [17],

Theorem 2.2. Suppose α_n and β_n are a Bailey pair relative to $(1, q)$ and $\alpha_0 = \beta_0 = 1$, then

$$\begin{aligned}
&\sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{(q; q)_{n_1}^2 q^{n_1+n_2+\dots+n_k} \beta_{n_1}}{(1 - q^{n_k})^2 (1 - q^{n_{k-1}})^2 \dots (1 - q^{n_1})^2} \\
&= \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{n_1+n_2+\dots+n_k}}{(1 - q^{n_k})^2 (1 - q^{n_{k-1}})^2 \dots (1 - q^{n_1})^2} + \sum_{r=1}^{\infty} \frac{q^{kr} \alpha_r}{(1 - q^r)^{2k}}.
\end{aligned}$$

The following is Corollary 3.4 of Theorem 3.3 from [17],

Corollary 2.3.

$$\sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{n_1+n_2+\dots+n_k}}{(1 - q^{n_k})^2 (1 - q^{n_{k-1}})^2 \dots (1 - q^{n_1})^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{n(n-1)/2+kn} (1 + q^n)}{(1 - q^n)^{2k}}.$$

For η_2 we'll use the following.

Corollary 2.4.

$$\begin{aligned}
&\sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{(q^2; q^2)_{n_1} q^{2n_1+2n_2+\dots+2n_k}}{(-q; q^2)_{n_1} (1 - q^{2n_k})^2 (1 - q^{2n_{k-1}})^2 \dots (1 - q^{2n_1})^2} \\
&= \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{2n_1+2n_2+\dots+2n_k}}{(1 - q^{2n_k})^2 (1 - q^{2n_{k-1}})^2 \dots (1 - q^{2n_1})^2} + \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(2n-1)+2kn} (1 + q^{2n})}{(1 - q^{2n})^{2k}}
\end{aligned}$$

Proof. We have a Bailey pair for $(1, q^2)$ from [23] given by

$$\begin{aligned}
\alpha_n &= \begin{cases} 1 & n = 0 \\ (-1)^n q^{2n^2} (q^n + q^{-n}) & n \geq 1 \end{cases} \\
\beta_n &= \frac{1}{(-q, q^2; q^2)_n}.
\end{aligned}$$

Applying Theorem 2.2 to this Bailey pair gives the identity. \square

For $\overline{\eta}$ we'll use the following.

Corollary 2.5.

$$\begin{aligned} & \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{(q; q)_{n_1}^2 q^{n_1+n_2+\dots+n_k}}{(q^2; q^2)_{n_1} (1-q^{n_k})^2 (1-q^{n_{k-1}})^2 \dots (1-q^{n_1})^2} \\ &= \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{n_1+n_2+\dots+n_k}}{(1-q^{n_k})^2 (1-q^{n_{k-1}})^2 \dots (1-q^{n_1})^2} + \sum_{n=1}^{\infty} \frac{(-1)^n 2q^{n^2+kn}}{(1-q^n)^{2k}} \end{aligned}$$

Proof. We have a Bailey pair for $(1, q)$ from [15] given by

$$\begin{aligned} \alpha_n &= \begin{cases} 1 & n = 0 \\ (-1)^n 2q^{n^2} & n \geq 1 \end{cases} \\ \beta_n &= \frac{1}{(q^2; q^2)_n}. \end{aligned}$$

Applying Theorem 2.2 to this Bailey pair gives the identity. \square

Lastly we'll use the following corollary for $\overline{\eta^2}$.

Corollary 2.6.

$$\begin{aligned} & \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{(q^2; q^2)_{n_1}^2 (q; q^2)_{n_1}^2 q^{2n_1+2n_2+\dots+2n_k}}{(q^2; q^2)_{2n_1} (1-q^{2n_k})^2 (1-q^{2n_{k-1}})^2 \dots (1-q^{2n_1})^2} \\ &= \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{2n_1+2n_2+\dots+2n_k}}{(1-q^{2n_k})^2 (1-q^{2n_{k-1}})^2 \dots (1-q^{2n_1})^2} + \sum_{n=1}^{\infty} \frac{(-1)^n 2q^{n^2+2kn}}{(1-q^{2n})^{2k}} \end{aligned}$$

Proof. As in the proof of Theorem 7 of [6], we have

$$\sum_{j=-L}^L \frac{z^j q^{j^2}}{(q^2; q^2)_{L-j} (q^2; q^2)_{L+j}} = \frac{(-zq, -q/z; q^2)_L}{(q^2; q^2)_{2L}}.$$

Setting $z = -1$ gives a Bailey pair relative to $(1, q^2)$ where α_n and β_n are

$$\begin{aligned} \alpha_n &= \begin{cases} 1 & n = 0 \\ (-1)^n 2q^{n^2} & n \geq 1 \end{cases} \\ \beta_n &= \frac{(q; q^2)_n^2}{(q^2; q^2)_{2n}}. \end{aligned}$$

Applying Theorem 2.2 to this Bailey pair gives the identity. \square

Next we find expressions for $\text{M2spt}_k(n)$, $\overline{\text{spt}}_k(n)$, and $\overline{\text{spt}2}_k(n)$.

Corollary 2.7. For all $k \geq 1$,

$$\begin{aligned} \sum_{n=1}^{\infty} \text{M2spt}_k(n) q^n &= \sum_{n=1}^{\infty} (\mu_{2k}(n) - \eta_{2k}(n)) q^n \\ &= \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{2n_1+2n_2+\dots+2n_k}}{(1-q^{2n_k})^2 (1-q^{2n_{k-1}})^2 \dots (1-q^{2n_1})^2} \frac{(-q^{2n_1+1}; q^2)_{\infty}}{(q^{2n_1+2}; q^2)_{\infty}} \end{aligned}$$

Proof. By Theorem 2.1, Corollary 2.3 with q replaced by q^2 , and Corollary 2.4, we have that

$$\begin{aligned}
& \sum_{n=1}^{\infty} (\mu_{2k}(n) - \eta_{2k}(n)) q^n \\
&= \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \sum_{n \geq 1} \frac{(-1)^{n+1} q^{n(n-1)+2kn} (1+q^{2n})}{(1-q^{2n})^{2k}} + \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \sum_{n \geq 1} \frac{(-1)^n q^{2n^2-n+2kn} (1+q^{2n})}{(1-q^{2n})^{2k}} \\
&= \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \left(\sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{2n_1+2n_2+\dots+2n_k}}{(1-q^{2n_k})^2 (1-q^{2n_{k-1}})^2 \dots (1-q^{2n_1})^2} + \sum_{n \geq 1} \frac{(-1)^n q^{2n^2-n+kn} (1+q^{2n})}{(1-q^{2n})^{2k}} \right) \\
&= \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{(q^2; q^2)_{n_1} q^{2n_1+2n_2+\dots+2n_k}}{(-q; q^2)_{n_1} (1-q^{2n_k})^2 (1-q^{2n_{k-1}})^2 \dots (1-q^{2n_1})^2} \\
&= \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{2n_1+2n_2+\dots+2n_k}}{(1-q^{2n_k})^2 (1-q^{2n_{k-1}})^2 \dots (1-q^{2n_1})^2} \frac{(-q^{2n_1+1}; q^2)_{\infty}}{(q^{2n_1+2}; q^2)_{\infty}}.
\end{aligned}$$

□

Corollary 2.8. For all $k \geq 1$,

$$\begin{aligned}
\sum_{n=1}^{\infty} \overline{\text{spt}}_k(n) q^n &= \sum_{n=1}^{\infty} (\overline{\mu}_{2k}(n) - \overline{\eta}_{2k}(n)) q^n \\
&= \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{n_1+n_2+\dots+n_k}}{(1-q^{n_k})^2 (1-q^{n_{k-1}})^2 \dots (1-q^{n_1})^2} \frac{(-q^{n_1+1}; q)_{\infty}}{(q^{n_1+1}; q)_{\infty}}
\end{aligned}$$

Proof. By Theorem 2.1, Corollary 2.3, and Corollary 2.5, we have that

$$\begin{aligned}
& \sum_{n=1}^{\infty} (\overline{\mu}_{2k}(n) - \overline{\eta}_{2k}(n)) q^n \\
&= \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \sum_{n \geq 1} \frac{(-1)^{n+1} q^{n(n-1)/2+kn} (1+q^n)}{(1-q^n)^{2k}} + 2 \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \sum_{n \geq 1} \frac{(-1)^n q^{n^2+kn}}{(1-q^n)^{2k}} \\
&= \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \left(\sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{n_1+n_2+\dots+n_k}}{(1-q^{n_k})^2 (1-q^{n_{k-1}})^2 \dots (1-q^{n_1})^2} + \sum_{n \geq 1} \frac{(-1)^n 2q^{n^2+kn}}{(1-q^n)^{2k}} \right) \\
&= \frac{(q^2; q^2)_{\infty}}{(q; q)_{\infty}^2} \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{(q; q)_{n_1} q^{n_1+n_2+\dots+n_k}}{(q^2; q^2)_{n_1} (1-q^{n_k})^2 (1-q^{n_{k-1}})^2 \dots (1-q^{n_1})^2} \\
&= \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{n_1+n_2+\dots+n_k}}{(1-q^{n_k})^2 (1-q^{n_{k-1}})^2 \dots (1-q^{n_1})^2} \frac{(q^{2n_1+2}; q^2)_{\infty}}{(q^{n_1+1}; q)_{\infty}^2} \\
&= \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{n_1+n_2+\dots+n_k}}{(1-q^{n_k})^2 (1-q^{n_{k-1}})^2 \dots (1-q^{n_1})^2} \frac{(-q^{n_1+1}; q)_{\infty}}{(q^{n_1+1}; q)_{\infty}}.
\end{aligned}$$

□

Corollary 2.9. For all $k \geq 1$,

$$\begin{aligned}
\sum_{n=1}^{\infty} \overline{\text{spt}}_{2k}(n) q^n &= \sum_{n=1}^{\infty} (\overline{\mu}_{2k}(n) - \overline{\eta}_{2k}(n)) q^n \\
&= \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{2n_1+2n_2+\dots+2n_k}}{(1-q^{2n_k})^2 (1-q^{2n_{k-1}})^2 \dots (1-q^{2n_1})^2} \frac{(-q^{2n_1+1}; q)_{\infty}}{(q^{2n_1+1}; q)_{\infty}}
\end{aligned}$$

Proof. By Theorem 2.1, Corollary 2.3 with q replaced by q^2 , and Corollary 2.6, we have that

$$\begin{aligned}
& \sum_{n=1}^{\infty} (\overline{\mu 2}_{2k}(n) - \overline{\eta 2}_{2k}(n)) q^n \\
&= \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \sum_{n \geq 1} \frac{(-1)^{n+1} q^{n(n-1)+2kn} (1+q^{2n})}{(1-q^{2n})^{2k}} + 2 \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \sum_{n \geq 1} \frac{(-1)^n q^{n^2+2kn}}{(1-q^{2n})^{2k}} \\
&= \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \left(\sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{2n_1+2n_2+\dots+2n_k}}{(1-q^{2n_k})^2 (1-q^{2n_{k-1}})^2 \dots (1-q^{2n_1})^2} + \sum_{n \geq 1} \frac{(-1)^n 2q^{n^2+2kn}}{(1-q^{2n})^{2k}} \right) \\
&= \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{(q^2; q^2)_{n_1}^2 (q; q^2)_{n_1}^2 q^{2n_1+2n_2+\dots+2n_k}}{(q^2; q^2)_{2n_1} (1-q^{2n_k})^2 (1-q^{2n_{k-1}})^2 \dots (1-q^{2n_1})^2} \\
&= \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}^2} \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{(q; q^2)_{2n_1}^2 q^{2n_1+2n_2+\dots+2n_k}}{(q^2; q^2)_{2n_1} (1-q^{2n_k})^2 (1-q^{2n_{k-1}})^2 \dots (1-q^{2n_1})^2} \\
&= \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{2n_1+2n_2+\dots+2n_k}}{(1-q^{2n_k})^2 (1-q^{2n_{k-1}})^2 \dots (1-q^{2n_1})^2} \frac{(q^{4n_1+2}; q^2)_{\infty}}{(q^{2n_1+1}; q^2)_{\infty}^2} \\
&= \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{2n_1+2n_2+\dots+2n_k}}{(1-q^{2n_k})^2 (1-q^{2n_{k-1}})^2 \dots (1-q^{2n_1})^2} \frac{(-q^{2n_1+1}; q)_{\infty}}{(q^{2n_1+1}; q)_{\infty}}.
\end{aligned}$$

□

It is now clear that $M2spt_k(n) \geq 0$, $\overline{spt}_k(n) \geq 0$, and $\overline{spt 2}_k(n) \geq 0$. Next we consider inequalities between the ordinary moments.

Corollary 2.10. *Suppose $k \geq 1$. For $n = 2$ and $n \geq 4$ we have*

$$M2_{2k}(n) > N2_{2k}(n).$$

For $n \geq 1$ we have

$$\overline{M}_{2k}(n) > \overline{N}_{2k}(n).$$

For $n = 2$ and $n \geq 4$ we have

$$\overline{M}2_{2k}(n) > \overline{N}2_{2k}(n).$$

Proof. We know

$$\sum_{n \geq 1} (\mu 2_{2j}(n) - \eta 2_{2j}(n)) q^n = \frac{q^{2j} (-q^3; q^2)_{\infty}}{(1-q^2)^{2j} (q^4; q^2)_{\infty}} + \dots$$

where the omitted terms also have non-negative coefficients. It is then apparent that

$$\mu 2_{2j}(n) > \eta 2_{2j}(n)$$

for $j \geq 1$ and $n \geq 2j + 2$. This inequality also holds when $n = 2j$, but we instead have equality at $2j + 1$.

However, the $S^*(k, j)$ are integers and are positive for $1 \leq j \leq k$, thus

$$\begin{aligned}
M2_{2k}(n) - N2_{2k}(n) &= \sum_{j=1}^k (2j)! S^*(k, j) (\mu 2_{2j}(n) - \eta 2_{2j}(n)) \\
&\geq \mu 2_2(n) - \eta 2_2(n) \\
&> 0,
\end{aligned}$$

for $n \geq 4$ and $n = 2$.

Next we have

$$\sum_{n \geq 1} (\bar{\mu}_{2j}(n) - \bar{\eta}_{2j}(n)) q^n = \frac{q^j (-q^2; q)_\infty}{(1-q)^{2j} (q^2; q)_\infty} + \dots$$

where the omitted terms also have non-negative coefficients. It is then apparent that

$$\bar{\mu}_{2j}(n) > \bar{\eta}_{2j}(n)$$

for $j \geq 1$ and $n \geq j$. Similar to the previous case,

$$\begin{aligned} \bar{M}_{2k}(n) - \bar{N}_{2k}(n) &= \sum_{j=1}^k (2j)! S^*(k, j) (\bar{\mu}_{2j}(n) - \bar{\eta}_{2j}(n)) \\ &\geq \bar{\mu}_2(n) - \bar{\eta}_2(n) \\ &> 0, \end{aligned}$$

for $n \geq 1$.

Last we have

$$\sum_{n \geq 1} (\bar{\mu}_{2j}(n) - \bar{\eta}_{2j}(n)) q^n = \frac{q^{2j} (-q^3; q)_\infty}{(1-q^2)^{2j} (q^3; q)_\infty} + \dots$$

where the omitted terms also have non-negative coefficients. Thus

$$\bar{\mu}_{2j}(n) > \bar{\eta}_{2j}(n)$$

for $j \geq 1$ and $n \geq 2j + 2$. This inequality also holds when $n = 2j$, but we instead have equality at $2j + 1$. As before

$$\begin{aligned} \bar{M}_{2k}(n) - \bar{N}_{2k}(n) &= \sum_{j=1}^k (2j)! S^*(k, j) (\bar{\mu}_{2j}(n) - \bar{\eta}_{2j}(n)) \\ &\geq \bar{\mu}_2(n) - \bar{\eta}_2(n) \\ &> 0, \end{aligned}$$

for $n \geq 4$ and $n = 2$. □

It is important to note that in [18] Larsen, Rust, and Swisher proved a stronger result than $\bar{M}_{2k}(n) > \bar{N}_{2k}(n)$. In particular they proved that

$$\bar{M}_k^+(n) > \bar{N}_k^+(n),$$

where

$$\begin{aligned} \bar{N}_k^+(n) &= \sum_{m \geq 1} m^k \bar{N}_2(m, n), \\ \bar{M}_k(n) &= \sum_{m \geq 1} m^k \bar{M}_2(m, n). \end{aligned}$$

The methods used to handle when the series are only over $m \geq 1$ are quite different than the methods used here. Also they extend a result of Mao [22] that $\bar{N}_{2k} > \bar{N}_{2k}^+$ to $\bar{N}_k^+ > \bar{N}_{2k}^+$. Mao's proof also used the expressions we've used for $\bar{\eta}_{2k}$ and $\bar{\eta}_{2k}^+$.

3. COMBINATORIAL INTERPRETATIONS

As in [17], for a partition π where the different parts are

$$n_1 < n_2 < \cdots < n_m,$$

we have $f_j = f_j(\pi)$ is the frequency of the part n_j .

Thinking of overpartition and partitions without repeated odd parts as pairs of partitions, we make the following definition. Suppose $\vec{\pi} = (\pi_1, \dots, \pi_r)$ is a vector partition of n , then $f_j^1 = f_j^1(\vec{\pi}) = f_j(\pi_1)$. We now view overpartitions as partition pairs where π_2 is a partition into distinct parts, and view partitions with distinct odd parts as partition pairs where π_1 has only even parts and π_2 has only distinct odd parts. For overpartitions with smallest part even, we use a slightly different idea. We view an overpartition with smallest part even as a vector partition $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$ where π_1 are the non-overlined even parts, π_2 are the non-overlined odd parts, and π_3 are the overlined parts. Furthermore, in all three cases we require the smallest part to only occur in π_1 . We denote the set of overpartitions with smallest part not overlined by $\overline{\text{S}}$, the set of partitions with smallest part even and non-repeated odds by S2 , and the set of overpartitions with smallest part even and not overlined by $\overline{\text{S2}}$.

We note that

$$\begin{aligned} \text{M2spt}(n) &= \sum_{\vec{\pi} \in \text{S2}, |\vec{\pi}|=n} f_1^1(\vec{\pi}), \\ \overline{\text{spt}}(n) &= \sum_{\vec{\pi} \in \overline{\text{S}}, |\vec{\pi}|=n} f_1^1(\vec{\pi}), \\ \overline{\text{spt2}}(n) &= \sum_{\vec{\pi} \in \overline{\text{S2}}, |\vec{\pi}|=n} f_1^1(\vec{\pi}). \end{aligned}$$

For $k \geq 1$ we extend the weight ω_k of [17], for a partition pair $\vec{\pi} = (\pi_1, \pi_2)$ or vector partition $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$ we let $\omega_k(\vec{\pi}) = \omega_k(\pi_1)$. That is,

$$\omega_k(\vec{\pi}) = \sum_{\substack{m_1+m_2+\dots+m_r=k \\ 1 \leq r \leq k}} \binom{f_1^1+m_1-1}{2m_1-1} \sum_{2 \leq j_2 < j_3 < \dots < j_r} \binom{f_{j_2}^1+m_2}{2m_2} \binom{f_{j_3}^1+m_3}{2m_3} \cdots \binom{f_{j_r}^1+m_r}{2m_r}.$$

Theorem 3.1. *For all $k \geq 1$ and $n \geq 1$ we have*

$$\begin{aligned} \text{M2spt}_k(n) &= \sum_{\vec{\pi} \in \text{S2}} \omega_k(\vec{\pi}), \\ \overline{\text{spt}}_k(n) &= \sum_{\vec{\pi} \in \overline{\text{S}}} \omega_k(\vec{\pi}), \\ \overline{\text{spt2}}_k(n) &= \sum_{\vec{\pi} \in \overline{\text{S2}}} \omega_k(\vec{\pi}). \end{aligned}$$

Proof. The proof is near identical as that of Theorem 5.6 of [17], the only difficulty being how to write out the general case. We will fully write out the case when $k = 3$ for $\overline{\text{spt}}_k(n)$, go over the case of $k = 4$ for $\text{M2spt}_k(n)$, and explain the procedure for general k which will then be clear.

We use

$$\begin{aligned} \sum_{n=j}^{\infty} \binom{n+j-1}{2j-1} x^n &= \frac{x^j}{(1-x)^{2j}} \\ \sum_{n=j}^{\infty} \binom{n+j}{2j} x^n &= \frac{x^j}{(1-x)^{2j+1}}. \end{aligned}$$

For the $k = 3$ case for $\overline{\text{spt}}_k(n)$, we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} (\overline{\mu}_6(n) - \overline{\eta}_6(n)) q^n \\
&= \sum_{1 \leq m \leq k \leq n} \frac{q^{m+k+n} (-q^{m+1}; q)_{\infty}}{(1-q^m)^2 (1-q^k)^2 (1-q^n)^2 (q^{m+1}; q)_{\infty}} \\
&= \left(\sum_{1 \leq m=k=n} + \sum_{1 \leq m=k < n} + \sum_{1 \leq m < k=n} + \sum_{1 \leq m < k < n} \right) \frac{q^{m+k+n} (-q^{m+1}; q)_{\infty}}{(1-q^m)^2 (1-q^k)^2 (1-q^n)^2 (q^{m+1}; q)_{\infty}} \\
&= \sum_{1 \leq m} \frac{q^{3m}}{(1-q^m)^6} (-q^{m+1}; q)_{\infty} \prod_{i>m} \frac{1}{1-q^i} + \sum_{1 \leq m < n} \frac{q^{2m}}{(1-q^m)^4} \frac{q^n}{(1-q^n)^3} (-q^{m+1}; q)_{\infty} \prod_{\substack{i>m \\ i \neq n}} \frac{1}{1-q^i} \\
&\quad + \sum_{1 \leq m < k} \frac{q^m}{(1-q^m)^2} \frac{q^{2k}}{(1-q^k)^5} (-q^{m+1}; q)_{\infty} \prod_{\substack{i>m \\ i \neq k}} \frac{1}{1-q^i} \\
&\quad + \sum_{1 \leq m < k < n} \frac{q^m}{(1-q^m)^2} \frac{q^k}{(1-q^k)^3} \frac{q^n}{(1-q^n)^3} (-q^{m+1}; q)_{\infty} \prod_{\substack{i>m \\ i \neq k, n}} \frac{1}{1-q^i} \\
&= \sum_{1 \leq m} \sum_{f_1=3}^{\infty} \binom{f_1+3-1}{6-1} q^{mf_1} (-q^{m+1}; q)_{\infty} \prod_{i>m} \frac{1}{1-q^i} \\
&\quad + \sum_{1 \leq m < n} \sum_{f_1=2}^{\infty} \binom{f_1+2-1}{4-1} q^{mf_1} \sum_{f_{j_2}=1}^{\infty} \binom{f_{j_2}+1}{2} q^{nf_{j_2}} (-q^{m+1}; q)_{\infty} \prod_{\substack{i>m \\ i \neq n}} \frac{1}{1-q^i} \\
&\quad + \sum_{1 \leq m < k} \sum_{f_1=1}^{\infty} \binom{f_1+1-1}{2-1} q^{mf_1} \sum_{f_{j_2}=2}^{\infty} \binom{f_{j_2}+2}{4} q^{kf_{j_2}} (-q^{m+1}; q)_{\infty} \prod_{\substack{i>m \\ i \neq k}} \frac{1}{1-q^i} \\
&\quad + \sum_{1 \leq m < k < n} \sum_{f_1=1}^{\infty} \binom{f_1+1-1}{2-1} q^{mf_1} \sum_{f_{j_2}=1}^{\infty} \binom{f_{j_2}+1}{2} q^{kf_{j_2}} \sum_{f_{j_3}=1}^{\infty} \binom{f_{j_3}+1}{2} q^{nf_{j_3}} (-q^{m+1}; q)_{\infty} \prod_{\substack{i>m \\ i \neq k, n}} \frac{1}{1-q^i}.
\end{aligned}$$

The set of the 4 compositions of 3 is $A = \{(3), (2, 1), (1, 2), (1, 1, 1)\}$, thus we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} (\overline{\mu}_6(n) - \overline{\eta}_6(n)) q^n \\
&= \sum_{(m_1, \dots, m_r) = \vec{m} \in A} \sum_{1 \leq n_1 < n_{j_2} < \dots < n_{j_r}} \sum_{f_1=m_1}^{\infty} \sum_{f_{j_2}=m_2}^{\infty} \dots \sum_{f_{j_r}=m_r}^{\infty} \binom{f_1+m_1-1}{2m_1-1} \binom{f_{j_2}+m_2}{2m_2} \dots \binom{f_{j_r}+m_r}{2m_r} \\
&\quad \times q^{n_1 f_1 + n_{j_2} f_{j_2} + \dots + n_{j_r} f_{j_r}} (-q^{n_1+1}; q)_{\infty} \prod_{\substack{i>n_1 \\ i \notin \{n_{j_2}, \dots, n_{j_r}\}}} \frac{1}{1-q^i}.
\end{aligned}$$

This we recognize as the generating function for partition pairs $\vec{\pi} = (\pi_1, \pi_2) \in \overline{\mathcal{S}}$ counted according to the weight ω_3 . This is the generating function obtained by summing according to the smallest part of π_1 being n_1 with frequency f_1 .

For the $k = 4$ case for $\text{M2spt}_k(n)$, we have

$$\sum_{n=1}^{\infty} (\mu_{28}(n) - \eta_{28}(n)) q^n = \sum_{1 \leq m \leq j \leq k \leq n} \frac{q^{2m+2j+2k+2n} (-q^{2m+1}; q^2)_{\infty}}{(1-q^{2m})^2 (1-q^{2j})^2 (1-q^{2k})^2 (1-q^{2n})^2 (q^{2m+2}; q^2)_{\infty}}$$

$$\begin{aligned}
&= \sum_{1 \leq m=j=k=n} + \sum_{1 \leq m=j=k < n} + \sum_{1 \leq m=j < k=n} + \sum_{1 \leq m=j < k < n} + \sum_{1 \leq m < j=k=n} + \sum_{1 \leq m < j=k < n} + \sum_{1 \leq m < j < k=n} \\
&\quad + \sum_{1 \leq m < j < k < n} \frac{q^{2m+2j+2k+2n} (-q^{2m+1}; q^2)_\infty}{(1-q^{2m})^2 (1-q^{2j})^2 (1-q^{2k})^2 (1-q^{2n})^2 (q^{2m+2}; q^2)_\infty} \\
&= \sum_{1 \leq m} \frac{q^{8m}}{(1-q^{2m})^8} (-q^{2m+1}; q^2)_\infty \prod_{i>m} \frac{1}{1-q^{2i}} \\
&\quad + \sum_{1 \leq m < n} \frac{q^{6m}}{(1-q^{2m})^6} \frac{q^{2n}}{(1-q^{2n})^3} (-q^{2m+1}; q^2)_\infty \prod_{\substack{i>m \\ i \neq n}} \frac{1}{1-q^{2i}} \\
&\quad + \sum_{1 \leq m < k} \frac{q^{4m}}{(1-q^{2m})^4} \frac{q^{4k}}{(1-q^{2k})^5} (-q^{2m+1}; q^2)_\infty \prod_{\substack{i>m \\ i \neq k}} \frac{1}{1-q^{2i}} \\
&\quad + \sum_{1 \leq m < k < n} \frac{q^{4m}}{(1-q^{2m})^4} \frac{q^{2k}}{(1-q^{2k})^3} \frac{q^{2n}}{(1-q^{2n})^3} (-q^{2m+1}; q^2)_\infty \prod_{\substack{i>m \\ i \neq k, n}} \frac{1}{1-q^{2i}} \\
&\quad + \sum_{1 \leq m < j} \frac{q^{2m}}{(1-q^{2m})^2} \frac{q^{6j}}{(1-q^{2j})^7} (-q^{2m+1}; q^2)_\infty \prod_{\substack{i>m \\ i \neq j}} \frac{1}{1-q^{2i}} \\
&\quad + \sum_{1 \leq m < j < n} \frac{q^{2m}}{(1-q^{2m})^2} \frac{q^{4j}}{(1-q^{2j})^5} \frac{q^{2n}}{(1-q^{2n})^3} (-q^{2m+1}; q^2)_\infty \prod_{\substack{i>m \\ i \neq j, n}} \frac{1}{1-q^{2i}} \\
&\quad + \sum_{1 \leq m < j < k} \frac{q^{2m}}{(1-q^{2m})^2} \frac{q^{2j}}{(1-q^{2j})^3} \frac{q^{4k}}{(1-q^{2k})^5} (-q^{2m+1}; q^2)_\infty \prod_{\substack{i>m \\ i \neq j, k}} \frac{1}{1-q^{2i}} \\
&\quad + \sum_{1 \leq m < j < k < n} \frac{q^{2m}}{(1-q^{2m})^2} \frac{q^{2j}}{(1-q^{2j})^3} \frac{q^{2k}}{(1-q^{2k})^3} \frac{q^{2n}}{(1-q^{2n})^3} (-q^{2m+1}; q^2)_\infty \prod_{\substack{i>m \\ i \neq j, k, n}} \frac{1}{1-q^{2i}}.
\end{aligned}$$

In order, the above eight terms correspond to the compositions of 4: (4), (3, 1), (2, 2), (2, 1, 1), (1, 3), (1, 2, 1), (1, 1, 2), (1, 1, 1, 1).

Thus for each composition $m_1 + \dots + m_r = 4$ we have a sum of the form:

$$\begin{aligned}
&\sum_{1 \leq n_1 < n_{j_2} < \dots < n_{j_r}} \frac{q^{2n_1 m_1}}{(1-q^{2n_1})^{2m_1}} \frac{q^{2n_2 m_2}}{(1-q^{2n_2})^{2m_2+1}} \dots \frac{q^{2n_{j_r} m_r}}{(1-q^{2n_{j_r}})^{2m_r+1}} (-q^{2n_1+1}; q^2)_\infty \prod_{\substack{i>n_1 \\ i \notin \{n_{j_2}, \dots, n_{j_r}\}}} \frac{1}{1-q^{2i}} \\
&= \sum_{1 \leq n_1 < n_{j_2} < \dots < n_{j_r}} \sum_{f_1=m_1}^\infty \sum_{f_{j_2}=m_2}^\infty \dots \sum_{f_{j_r}=m_r}^\infty \binom{f_1+m_1-1}{2m_1-1} \binom{f_{j_2}+m_2}{2m_2} \dots \binom{f_{j_r}+m_r}{2m_r} \\
&\quad \times q^{2n_1 f_1 + 2n_{j_2} f_{j_2} + \dots + 2n_{j_r} f_{j_r}} (-q^{2n_1+1}; q^2)_\infty \prod_{\substack{i>n_1 \\ i \notin \{n_{j_2}, \dots, n_{j_r}\}}} \frac{1}{1-q^{2i}}. \tag{3.1}
\end{aligned}$$

Noting the f_{j_i} correspond to the frequencies of certain even parts, we see summing (3.1) over all compositions of 4 yields the generating function for partitions without repeated odd parts and smallest part even written as a partition pair $(\pi_1, \pi_2) \in S2$, counted according to the weight ω_4 . This is the generating function given by summing according to the smallest part being $2n_1$ with frequency f_1 .

For general k , we take the expression in Corollary 2.7, 2.8, or 2.9 and break it into 2^{k-1} sums by turning the index bounds into $=$ or $<$. These correspond to the 2^{k-1} compositions of k . The sum with index bounds $n_1 \square_1 n_2 \square_2 \dots \square_{r-1} n_r$ where each \square_i is either “=” or “<” corresponds to the composition $(1\triangle_1 1\triangle_2 \dots \triangle_{r-1} 1)$ where \triangle_i is “+” if \square_i is “=” and \triangle_i is “,” if \square_i is “<”.

For $\text{M2spt}_k(n)$ the sum corresponding to the fixed composition $m_1 + m_2 + \cdots + m_r = k$ is then rewritten as

$$\begin{aligned} & \sum_{1 \leq n_1 < n_{j_2} < \cdots < n_{j_r}} \sum_{f_1=m_1}^{\infty} \sum_{f_{j_2}=m_2}^{\infty} \cdots \sum_{f_{j_r}=m_r}^{\infty} \binom{f_1+m_1-1}{2m_1-1} \binom{f_{j_2}+m_2}{2m_2} \cdots \binom{f_{j_r}+m_r}{2m_r} \\ & \times q^{2n_1 f_1 + 2n_{j_2} f_{j_2} + \cdots + 2n_{j_r} f_{j_r}} (-q^{2n_1+1}; q^2)_{\infty} \prod_{\substack{i > n_1 \\ i \notin \{n_{j_2}, \dots, n_{j_r}\}}} \frac{1}{1-q^{2i}} \end{aligned} \quad (3.2)$$

Thus on the one hand summing (3.2) over all compositions of k gives $\sum_{n=1}^{\infty} (\mu_{2k}(n) - \eta_{2k}(n)) q^n$, but also this is $\sum_{n=1}^{\infty} q^n \sum_{\vec{\pi} \in \text{S2}} \omega_k(\vec{\pi})$.

For $\overline{\text{spt}}_k(n)$, the general case follows the same idea, but differs in that the term for a fixed composition $m_1 + \cdots + m_r$ of k is

$$\begin{aligned} & \sum_{1 \leq n_1 < n_{j_2} < \cdots < n_{j_r}} \sum_{f_1=m_1}^{\infty} \sum_{f_{j_2}=m_2}^{\infty} \cdots \sum_{f_{j_r}=m_r}^{\infty} \binom{f_1+m_1-1}{2m_1-1} \binom{f_{j_2}+m_2}{2m_2} \cdots \binom{f_{j_r}+m_r}{2m_r} \\ & \times q^{n_1 f_1 + n_{j_2} f_{j_2} + \cdots + n_{j_r} f_{j_r}} (-q^{n_1+1}; q)_{\infty} \prod_{\substack{i > n_1 \\ i \notin \{n_{j_2}, \dots, n_{j_r}\}}} \frac{1}{1-q^i}. \end{aligned}$$

Lastly, for $\overline{\text{spt}}_2(n)$ the general term is

$$\begin{aligned} & \sum_{1 \leq n_1 < n_{j_2} < \cdots < n_{j_r}} \sum_{f_1=m_1}^{\infty} \sum_{f_{j_2}=m_2}^{\infty} \cdots \sum_{f_{j_r}=m_r}^{\infty} \binom{f_1+m_1-1}{2m_1-1} \binom{f_{j_2}+m_2}{2m_2} \cdots \binom{f_{j_r}+m_r}{2m_r} \\ & \times q^{2n_1 f_1 + 2n_{j_2} f_{j_2} + \cdots + 2n_{j_r} f_{j_r}} (-q^{2n_1+1}; q)_{\infty} \prod_{\substack{i > 2n_1 \\ i \notin \{2n_{j_2}, \dots, 2n_{j_r}\}}} \frac{1}{1-q^i}. \end{aligned}$$

This finishes the proof. \square

4. CONGRUENCES FOR $\overline{\text{spt}}_2(n)$

It appears that these higher order spt functions satisfy various congruences. We prove two of them.

Theorem 4.1. For $n \geq 0$,

$$\begin{aligned} \overline{\text{spt}}_2(5n+1) &\equiv 0 \pmod{5}, \\ \overline{\text{spt}}_2(5n+3) &\equiv 0 \pmod{5}. \end{aligned}$$

Proof. We have

$$\overline{\text{spt}}_2(n) = \frac{1}{24} (\overline{M}_4(n) - \overline{M}_2(n) - \overline{N}_4(n) + \overline{N}_2(n)). \quad (4.1)$$

Reducing equation (3.1) of [9] modulo 5 gives

$$\overline{N}_4(n) \equiv (2n+4)\overline{N}_2(n) + (2n+2)\overline{M}_2(n) + \overline{M}_4(n) + 2n\overline{M}_2(n) \pmod{5},$$

so (4.1) becomes

$$\overline{\text{spt}}_2(n) \equiv (3+2n)\overline{M}_2(n) + 2n\overline{M}_2(n) + (3+2n)\overline{N}_2(n) \pmod{5}. \quad (4.2)$$

The following are equation (4.4) and an equation out of the proof of Theorem 3.1 of [9]:

$$(2n^2 + n + 2)\overline{M}_2(n) + (n^2 + 4n + 2)\overline{M}_2(n) \equiv 0 \pmod{5}, \quad (4.3)$$

$$(4n^2 + n)\overline{M}_2(n) + (4n+4)\overline{M}_2(n) + (3n^2 + 2)\overline{N}_2(n) \equiv 0 \pmod{5}. \quad (4.4)$$

In (4.3) we replace n by $5n + 1$ and in (4.4) we replace n by $5n + 3$ to get

$$\begin{aligned}\overline{M2}_2(5n + 1) &\equiv 0 \pmod{5}, \\ 4\overline{M}_2(5n + 3) + \overline{M2}_2(5n + 3) + 4\overline{N}_2(5n + 3) &\equiv 0 \pmod{5}.\end{aligned}$$

With (4.2) we then have

$$\begin{aligned}\overline{\text{spt}}_2(5n + 1) &\equiv 2\overline{M2}_2(5n + 1) \equiv 0 \pmod{5}, \\ \overline{\text{spt}}_2(5n + 3) &\equiv 4\overline{M}_2(5n + 3) + 4\overline{N}_2(5n + 3) + \overline{M2}_2(5n + 3) \equiv 0 \pmod{5}.\end{aligned}$$

□

5. REMARKS

In [13] Dixit and Yee also generalized the spt function to Spt_j and generalized the higher order spt -function spt_k to ${}_j\text{spt}_k$. They used

$$\begin{aligned}\text{Spt}_j(n) &= \frac{1}{2}M_2(n) - \frac{1}{2^{j+1}}N_2(n), \\ {}_j\text{spt}_k(n) &= {}_j\mu_{2k}(n) - {}_{j+1}\mu_{2k}(n),\end{aligned}$$

where

$$\begin{aligned}{}_jN_k(n) &= \sum_{m \in \mathbb{Z}} m^k N_j(m, n), \\ {}_j\mu_k(n) &= \sum_{m \in \mathbb{Z}} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} N_j(m, n),\end{aligned}$$

and $N_j(m, n)$ is the number of partitions of n with at least $j - 1$ successive Durfee squares whose j -rank is m . It may be possible to work out generalizations of this form for the three spt functions we've investigated here.

It is worth mentioning that it is not $\overline{R2}$ and $\overline{C2}$ that were used in [15] to reprove certain congruences satisfied by $\overline{\text{spt}2}(n)$. However, the methods in that paper can be used with $\overline{R2}$ and $\overline{C2}$ to prove the congruences $\overline{\text{spt}2}(3n) \equiv \overline{\text{spt}2}(3n + 1) \equiv 0 \pmod{3}$. Yet those methods do no work to prove the congruence $\overline{\text{spt}2}(5n + 3) \equiv 0 \pmod{5}$ with $\overline{R2}$ and $\overline{C2}$.

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